There are only a few Padovan squares in Padua

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Padovan squares in Padua

The Padovan numbers P_n are for $n = 0, 1, 2, \ldots$ defined by the Fibonacci-like recurrence relation $P_{n+1} = P_{n-1} + P_{n-2}$, with the initial values $P_0 = P_1 = P_2 = 1$. The first few values are $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, \ldots$.

In Chapter Eight of his book Math Hysteria [S2], and originally in his Scientific American column [S1], Ian Stewart asks if there are any Padovan numbers beyond $P_{15} = 49$ that are squares. This seems to be a hard problem. He also notes that the squares that do occur in this small list are squares of Padovan numbers themselves, and he asks if that is always the case. That also seems a hard problem, though clearly not harder than the first one.

Surprisingly, Stewart's two problems are equivalent. This follows from the fact that there are no Padovan numbers beyond $P_{15} = 49$ that are squares of Padovan numbers. The purpose of this note is to prove that fact, and I do so in a completely elementary way.

The result

Theorem *The only solutions of $P_n = P_m^2$ in integers n, m larger than 2 are $P_6 = P_3^2 = 4$, $P_6 = P_4^2 = 4$, $P_9 = P_5^2 = 9$, $P_{11} = P_6^2 = 16$, and $P_{15} = P_8^2 = 49$.*

The proof

Let $\alpha = 1.3247\ldots$ be the real root of $x^3 - x - 1$, and let $\beta, \overline{\beta}$ be the two non-real conjugates of α. Let $\lambda = \frac{1}{23}(3 + \alpha + 7\alpha^2) = 0.7721\ldots$, and $\mu = \frac{1}{23}(3 + \beta + 7\beta^2)$. Then $|\beta| = \alpha^{-\frac{1}{2}}$ and $|\mu| = \frac{1}{\sqrt{23}}\lambda^{-\frac{1}{2}}$. It is not hard to show that $P_n = \lambda\alpha^n + \mu\beta^n + \overline{\mu}\overline{\beta}^n$ holds for all n. Clearly

$$|P_n - \lambda\alpha^n| \le 2|\mu||\beta|^n = \frac{2}{\sqrt{23}}\lambda^{-\frac{1}{2}}\alpha^{-\frac{1}{2}n},$$

which tends to 0 if n grows. This simple fact is the heart of the proof.

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Indeed, $P_n = P_m^2$ implies

$$
\begin{aligned}
|\lambda\alpha^n - \lambda^2\alpha^{2m}| &\leq |\lambda\alpha^n - P_n| + |P_m^2 - \lambda^2\alpha^{2m}| \\
&\leq |\lambda\alpha^n - P_n| + |P_m - \lambda\alpha^m||P_m + \lambda\alpha^m| \\
&\leq |\lambda\alpha^n - P_n| + |P_m - \lambda\alpha^m|(2\lambda\alpha^m + |P_m - \lambda\alpha^m|) \\
&\leq \frac{2}{\sqrt{23}}\lambda^{-\frac{1}{2}}\alpha^{-\frac{1}{2}n} + \frac{4}{\sqrt{23}}\lambda^{\frac{1}{2}}\alpha^{\frac{1}{2}m} + \frac{4}{23}\lambda^{-1}\alpha^{-m},
\end{aligned}
$$

hence

$$
|\alpha^{n-2m} - \lambda| \leq \frac{2}{\sqrt{23}}\lambda^{-\frac{3}{2}}\alpha^{-\frac{1}{2}n-2m} + \frac{4}{\sqrt{23}}\lambda^{-\frac{1}{2}}\alpha^{-\frac{3}{2}m} + \frac{4}{23}\lambda^{-2}\alpha^{-3m}. \tag{1}
$$

The left hand side of (1) is minimal for $n - 2m = -1$, namely

$$
|\alpha^{n-2m} - \lambda| \geq |\alpha^{-1} - \lambda| = 0.03275\ldots.
$$

When $m \geq 9$ the right hand side of (1) is smaller than that, namely

$$
\frac{2}{\sqrt{23}}\lambda^{-\frac{3}{2}}\alpha^{-\frac{1}{2}n-2m} + \frac{4}{\sqrt{23}}\lambda^{-\frac{1}{2}}\alpha^{-\frac{3}{2}m} + \frac{4}{23}\lambda^{-2}\alpha^{-3m} \leq 0.02651\ldots.
$$

It follows that $m \leq 8$. Indeed, inequality (1) is valid only for $(n, m) = (15, 8), (13, 7), (11, 6), (9, 5)$ and some more values with $n \leq 7$. The result now follows by inspection.

Remarks

Note that in [dW] I did solve the problem, posed by Ian Stewart in [S1] and mentioned in [S2], which Padovan numbers are Fibonacci numbers (in fact I showed much more, namely that the distances between Padovan and Fibonacci numbers grow exponentially).

Extending the Padovan sequence to negative indices, the following problems can be stated: solve $P_{-n} = \pm P_{-m}^2$, solve $P_{-n} = \pm P_m^2$, and solve $P_n = P_{-m}^2$, all for positive integers n, m. I believe that these problems will be a lot harder.

References

[S1]   IAN STEWART, "Mathematical Recreations: Tales of a Neglected Number", *Scientific American* **274** [1996], 92–93.

[S2]   IAN STEWART, *Math Hysteria*, Oxford University Press, 2004.

[dW]   BENJAMIN M.M. DE WEGER, "Padua and Pisa are exponentially far apart", *Publicacions Matemàtiques (Barcelona)* **41** [1997], 631–651.